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Prediction Error based Interaction Measure for Control Configuration Selection in Linear and Nonlinear Systems [1]

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Abstract: This paper introduces an Interaction Measure named Prediction Error Index Array (PEIA), which can be applied both to linear and non-linear systems. The linear PEIA is constructed as an extension of previous results using the $\mathcal{H}_2$-norm. The non-linear PEIA is an extension for systems represented by Volterra series. Additionally, the paper gives an interpretation of both linear and nonlinear PEIA based on the prediction error of the structurally reduced model which results from the control configuration selection. Examples illustrate and compare the interaction measure with established methodologies, like the Relative Gain Array, participation matrix, and Hankel Interaction Index array.

1. INTRODUCTION

Prior to the synthesis of controller parameters in a multivariable process, a low complexity control configuration is often selected in a step known as Control Configuration Selection (CCS). One approach is to compose a structurally reduced model by selecting the most important input-output interconnections of the complete model. These interconnections should be considered in the design of the closed loop system, while the others can be neglected. This selection is often performed with the use of Interaction Measures (IMs), which include the modern gramian-based IMs (Salgado and Conley, 2004). It is well understood that system gramians can be used to identify the most significant portions of a system model.

Similarly, system gramians have been used for model reduction, which aims at simplifying complex dynamic models while appropriately representing the system behavior (Schilders et al., 2008). The difference between the output of the original model and the output of the structurally reduced model (denoted as model error), can be treated as noise and should be kept small for a well defined class of input signals.

Clearly, the CCS and the model reduction problems are related. One difference is that CCS often considers several different criteria based on controllability and observability, whilst model reduction is focused on minimizing properties of the prediction error. A natural next step would be to understand if and how methodologies from model reduction can be introduced in CCS.

In this paper, the CCS for linear and nonlinear systems is formulated based on analysis of the prediction error and controllability analysis, therefore relating to the model reduction problem. The presented results on linear systems are based on the use of the $\mathcal{H}_2$-norm, which was first used in the gramian-based IM known as Σ_2 for quantifying the output controllability (Birk and Medvedev, 2003).

While there is vast host of IMs in the linear framework, the amount of methods for nonlinear models is still rather limited. A typical approach is to apply IMs on a linearized model for specific operating conditions. Alternatively, nonlinearities can often be considered as unmodelled dynamics and captured by an uncertainty description which is incorporated in a robust CCS method Castaño and Birk (2015b). However, the number of robust CCS methods is still limited, complex computations are required, and the decisions are often conservative (Castaño and Birk, 2016).

The main contribution of this paper is to extend the concept of the prediction error analysis to nonlinear systems, in order to introduce a new IM for models represented by Volterra series. Volterra series are often used to generalize concepts for their application on nonlinear systems (Volterra, 2005). For example, Wiener and Hammerstein systems can be precisely represented by Volterra series, and the more general modulator-demodulator systems can also be represented by Volterra series through the use of power series expansions (Bedrosian and Rice, 1971).

This paper is structured as follows. In Section 2, the Prediction Error Index Array (PEIA) is introduced. Section 3 introduces an extension of the PEIA for its application on nonlinear systems. Later, Section 4 compares the linear and nonlinear versions of PEIA with previously existing IMs. Finally, the conclusions are given in Section 5.

2. LINEAR INTERACTION MEASURE BASED ON THE PREDICTION ERROR

Preliminaries on linear systems are first given, followed by the introduction of the Prediction Error Index Array (PEIA). Later, relationships of PEIA with absolute and relative measures of the prediction error are derived.

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2.1 System gramians and the $\mathcal{H}_2$-norm

Consider the inear process with n inputs and m outputs:

$$\dot{x}(t) = Ax(t) + Bu(t)\ ;\ y(t) = Cx(t)$$

where $u \in \mathcal{R}^{n\times 1}$, $y \in \mathcal{R}^{m\times 1}$ and $x \in \mathcal{R}^{p\times 1}$ are the input, output and state vectors. The process can alternatively be represented by the transfer function $G(s) = C(sI - A)^{-1}B$ or by the impulse response $g(t)$, which is the inverse Laplace transform of $G(s)$.

Σ_2 is an IM introduced by Birk and Medvedev (2003) as:

$$[\Sigma_2]_{ij} = \frac{\|G_{ij}\|_2}{\sum\limits_{k,l=1}^{m,n} \|G_{kl}\|_2}$$

where $\|G_{ij}\|_2$ is the $\mathcal{H}_2$-norm of $G_{ij}(s)$.

Different ways to calculate of the $\mathcal{H}_2$-norm are:

$$\|G_{ij}\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |G_{ij}(\mathrm{j}\omega)|^2 d\omega} = \sqrt{\int_0^{2\pi} g_{ij}^2(\tau)d\tau}$$
$$= \sqrt{trace(B_j^T Q_i B_j)} = \sqrt{trace(C_i P_j C_i^T)} \qquad (1)$$

where $P_j = \int_0^\infty e^{A\tau}B_jB_j^Te^{A^T\tau}d\tau$ is the controllability gramian related to the j-th input, $Q_i = \int_0^\infty e^{A^T\tau}C_i^TC_ie^{A\tau}d\tau$ is the obsevabillity gramian related to the i-th output, B_j is the j-th row of B, and C_i is the i-th row of C.

Eq. 1 leads to different interpretation of the squared $\mathcal{H}_2$-norm as: (1) the output energy when the input is exited with an input signal with unitary flat power spectral density (psd), (2) the energy of the impulse response of the system, (3) a quantification of output controllability (Birk and Medvedev, 2003). These interpretations indicate the square of the $\mathcal{H}_2$-norm which will be used by PEIA is a more sensitive measure than the direct use of the $\mathcal{H}_2$-norm by Σ_2 in Eq. (1).

2.2 Definition of the linear Prediction Error Index Array.

We adapt the original definition of Σ_2, and define an IM named Prediction Error Index Array (PEIA) as:

$$[PEIA]_{ij} \triangleq \frac{\|G_{ij}\|_2^2}{\sum\limits_{k,l=1}^{m,n} \|G_{kl}\|_2^2} = \frac{\|G_{ij}\|_2^2}{\|G\|_2^2}$$

The name PEIA refers to the direct relationship of each element with the prediction error committed when neglecting the corresponding input-output channel. This relationship will be proven in Subsection 2.4.

In addition to the more direct interpretations than Σ_2, the sum of the individual metrics of the input-output channels in PEIA is equal to the metric of the complete system:

$$\sum \|G_{ij}\|_2^2 = \|G\|^2$$

This desirable property is a consequence of the gramian decomposition, and therefore the elements in PEIA express the contribution of each input-output channel as a fraction of the global contribution. This property is preserved by the first introduced gramian-based IMs named Participation Matrix (Salgado and Conley, 2004) and Hankel Interaction Index Array (Wittenmark and Salgado, 2002), but not by Σ_2.

The resulting configurations in the following examples are related to the simplest structurally reduced model with a contribution larger than 70%. This threshold on the contribution has to be adapted depending on the size of the system (Salgado and Conley, 2004). More details can be found in the literature on procedures and rules to follow during the selection of a control configuration from an gramian-based IA (Salgado and Conley, 2004; Castaño and Birk, 2016).

2.3 Absolute measure of the prediction error.

Denote by:

- $\hat{G}(\omega)$ the structurally reduced model on which control will be based.
- $\Delta G(\omega)$ the model composed by the disregarded IO channels.
- $\hat{y}(t) \in \mathcal{R}^{m,1}$ the output from the structurally reduced model $\hat{G}$.
- $y_\Delta(t)\mathcal{R}^{m,1}$ the prediction error, which is the output from the model ΔG.

Lemma 1. The squared $\mathcal{H}_2$-norm of the model ΔG is the average power of the prediction error of the structurally reduced model $\hat{G}(\omega)$ when the input signals are uncorrelated and have flat unitary psd.

Proof: The prediction error is defined as the difference $y_\Delta = y - \hat{y}$, and its average power is

$$P(y(t) - \hat{y}(t)) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} y_\Delta^T \cdot y_\Delta dt = \int_{-\infty}^{\infty} trace(S_{y_\Delta y_\Delta}(f))df$$

where $S_{y_\Delta y_\Delta} \in \mathcal{R}^{m,m}$ is the power spectral density (psd) of the prediction error $y_\Delta(t)$. The psd of the output of a linear system can be expressed as a function of the psd of its input S_{uu}, leading to

$$P(y(t) - \hat{y}(t)) = \int_{-\infty}^{\infty} trace(\Delta G(-f)\cdot S_{uu}(f)\cdot \Delta G(f)^T)df$$

Assuming that $u_i(t)$ are uncorrelated sequences with flat unitary psd, then $S_{uu}(f) = I$:

$$P(y(t) - \hat{y}(t)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} trace(\Delta G(-\omega)\cdot \Delta G(\omega)^T)d\omega$$
$$= \|\Delta G\|_2^2 = \sum_{i,j} \|\Delta G_{ij}\|_2^2$$

■

For the case of continuous-time systems, an input signal with flat unitary psd over all frequencies is not realizable since it has infinite energy. A band limited noise with flat band in an interval $[a, b]$ can be used, leading to the following integral:

$$P(y(t) - \hat{y}(t)) = \frac{1}{\pi}\int_a^b trace(\Delta G(-\omega)\cdot \Delta G(\omega)^T)d\omega$$

which equals the frequency-limited $\mathcal{H}_2$-norm (Vuillemin et al., 2014) used by Castaño and Birk (2015a) to restrict CCS to a range of frequencies.

2.4 Relative measure of the prediction error

Denote by:

- $\hat{\Omega} = \{(i,j) : G_{ij} \in \hat{G}(\omega)\}$
- $\Delta\Omega : \{(i,j) : G_{ij} \in \Delta G(\omega)\}$

Lemma 2. The sum of the elements of the PEIA (with indexes belonging to $\Delta\Omega$) which correspond to the neglected

I/0 channels in the structurally reduced model ΔG is a relative measure of the average power of the prediction error of the structurally reduced model $\hat{G}$ when the inputs are uncorrelated zero mean processes with flat psd.

Proof: Start by relating the power of the output y to the power of $\hat{y}$ and y_Δ:

$$P(y(t)) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}(y_\Delta(t)+\hat{y}(t))^T\cdot(y_\Delta(t)+\hat{y}(t))dt$$
$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\left(y_\Delta(t)^T y_\Delta(t)+\hat{y}(t)^T\hat{y}+2y_\Delta(t)^T\hat{y}(t)\right)dt$$

Each of the outputs $y_\Delta(t)$ and $\hat{y}(t)$ from ΔG and $\hat{G}$, are zero mean stochastic processes since they are the outputs of linear systems having zero mean stochastic inputs. Additionally, $[y_\Delta(t)]_i$ and $[\hat{y}(t)]_i$ are clearly uncorrelated, since the structures of ΔG and G are complementary, and therefore $[y_\Delta(t)]_i$ and $[\hat{y}(t)]_i$ have contributions from different inputs. Therefore, $E([y_\Delta(t)]_i\cdot[\hat{y}(t)]_i)=0$ and

$$P(y(t)) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\left(y_\Delta(t)^T y_\Delta(t)+\hat{y}(t)^T\hat{y}(t)\right)dt$$
$$= P(y_\Delta(t)) + P(\hat{y}(t))$$

The average power of the output is thus the power of the output from the structurally reduced model plus the average power of the output from the disregarded channels ΔG. A relative measure of the prediction error is,

$$\frac{P(y(t)-\hat{y}(t))}{P(y(t))} = \frac{\|\Delta G\|_2^2}{\|G\|_2^2} = \sum_{i,j}\frac{\|\Delta G_{ij}\|_2^2}{\|G\|_2^2} = \sum_{(i,j)\in\Delta\Omega}[PEIA]_{ij} \tag{2}$$

■

Another relevant measure is the ratio of the power of the structurally reduced model relative to the output of the original model:

$$\frac{P(\hat{y}(t))}{P(y(t))} = \frac{P(y(t))-P(y(t)-\hat{y}(t))}{P(y(t))} = \frac{\sum_{i,j}\|G_{ij}\|_2^2 - \sum_{i,j}\|\Delta G_{ij}\|_2^2}{\|G\|_2^2}$$
$$= \sum_{i,j}\frac{\|\hat{G}_{ij}\|_2^2}{\|G\|_2^2} = \sum_{(i,j)\in\hat{\Omega}}[PEIA]_{ij} = 1 - \sum_{(i,j)\in\Delta\Omega}[PEIA]_{ij} \tag{3}$$

Example 1.
Consider a process represented by the following multivariable transfer function:

$$G(s) = \begin{pmatrix} \frac{2}{(s+1)(s+2)} & \frac{-0.8s+0.55}{(s+5)(s+2)} & \frac{-0.5}{(s+4)} \\ \frac{2}{(s^2+3s+20)} & \frac{2.4}{(s^2+2s+4)} & \frac{0.5(-3.5s+1)}{(s+4)(s+5)} \\ \frac{0.5}{(s+2)} & \frac{3}{(s+3)^2} & \frac{6}{(s+2)(s+5)} \end{pmatrix} \tag{4}$$

The calculation of PEIA results in:

$$PEIA = \begin{pmatrix} \underline{0.2416} & 0.0347 & 0.0227 \\ 0.0242 & \underline{0.2609} & \underline{0.1238} \\ 0.0453 & 0.0604 & \underline{0.1864} \end{pmatrix} \tag{5}$$

$$PEIA_{11} + PEIA_{22} + PEIA_{33} + PEIA_{23} = 0.8128 \tag{6}$$
$$PEIA_{11} + PEIA_{22} + PEIA_{33} = 0.6890 \tag{7}$$

The simplest structurally reduced model $\hat{G}$ with a contribution larger than 0.7 (see Eq. (6)) is composed by the input-output channels: $\{(1,1),(2,2),(3,3),(2,3)\}$. According to Lemma 1, this structurally reduced model has a prediction error of approximately $(1-0.8128)\cdot 100\% = 18.72\%$ measured in terms of output power under an uncorrelated excitation sequence with flat psd. According to Eq. (7), a diagonal decentralized configuration would be related to a prediction error of $(1-0.6890)\cdot 100\% = 31.1\%$. The contribution of the decentralized configuration is close to the designed threshold of 0.7, which indicates that it is appropriate to test a simple diagonal decentralized configuration and, if the resulting performance is not satisfactory, use the sparse configuration related to Eq. (6).

3. INTERACTION MEASURE FOR NONLINEAR SYSTEMS BASED ON THE PREDICTION ERROR

In this section, preliminaries on Volterra Series are given, followed by the calculation of the contribution from each input to the output variance. This calculation is later used to define the PEIA for nonlinear systems.

3.1 Introduction to Volterra series.

Subsection 3.1 and Subsection 3.2 consider Single-Input and Single-Output nonlinear systems represented by:

$$y(t) = \mathcal{H}[u(t)]$$

If the operator $\mathcal{H}[\cdot]$ is time-invariant and has finite memory, its output can be expressed through the Volterra-series expansion given by (Schetzen, 2006):

$$y(t) = \sum_{k=0}^{\infty}\mathcal{H}^{(k)}[u(t)]$$

where $\mathcal{H}^{(k)}[]$ is the k-th order Volterra operator. The term $\mathcal{H}^{(0)}$ is a constant output independent of the input, while the rest of the terms are given by:

$$\mathcal{H}^{(k)}[u(t)] = \int_{\boldsymbol{\tau}_k\in\mathcal{R}^k} h^{(k)}(\boldsymbol{\tau}_k)\prod_{r=1}^{k}u(t-\tau_r)d\tau_k \quad (k=1,2,\dots)$$

where $\boldsymbol{\tau}_k = [\tau_1,\dots,\tau_k]^T$ contains the k integration variables, and the functions $h^{(k)}(\boldsymbol{\tau}_k)$ are the Volterra kernels. The first order term is the convolution integral typical of a linear dynamic system with $h^{(1)}(\tau_1)$ being the impulse response function. The higher order terms are multiple convolutions, involving products of the input values for different time delays. The expanded version of this equation is given in the Table 1 for different kernel orders.

An alternative representation in the frequency domain is provided by the Volterra Frequency Response Function (VFRF), which is the multidimensional Fourier transform of the Volterra kernels, i.e.

$$H^{(k)}(\boldsymbol{\Omega}_k) = \int_{\boldsymbol{\tau}_k\in\mathcal{R}^k} e^{-\boldsymbol{j}\boldsymbol{\Omega}_k^T\boldsymbol{\tau}_k}\cdot h^{(k)}(\boldsymbol{\tau}_k)\cdot d\boldsymbol{\tau}_k\ ;\ k=(1,2,3,\dots)$$

where $\boldsymbol{\Omega}_k = [\omega_1,\dots,\omega_k]^T$ and $H^{(0)} = \mathcal{H}^{(0)}$. In the sequel, we assume that the kernels represented by $h^{(k)}(\boldsymbol{\tau}_k)$ or $H^{(k)}(\boldsymbol{\Omega}_k)$ are symmetric with respect to permutations in the variables of the vectors $\boldsymbol{\tau}_k$ or $\boldsymbol{\Omega}_k$ respectively. Methods for the symmetrization of kernels are available in the literature (Mathews and Sicuranza, 2000).

3.2 Calculating the output variance for nonlinear systems.

The variance of the output variance can be calculated as:

$$\sigma_y^2 = \underbrace{E(y^2(t))}_{P(y(t))} - \underbrace{[E(y(t))]^2}_{P_{DC}(y(t))} \tag{8}$$

where $P_{DC}(y(t))$ is a DC term in the power which is generated by kernels of even index. The total power $P(y(t))$ is calculated integrating the output psd $S_{yy}(\omega)$:

$$P(y(t)) = E(y^2(t)) = \int_{-\infty}^{\infty}S_{yy}(\omega)d\omega$$

where

$$S_{yy}(\omega) = \frac{1}{(2\pi)^{n-1}} \sum_p \int_{-\infty}^{\infty} \left[H(\omega_1,\ldots,\omega_n)\cdot H(\omega_{n+1},\ldots,\omega_{2n})\right]_{sym} \cdot\delta_0(\omega-\omega_1-\cdots-\omega_n)\cdot\prod_{j,k}^{2n} S_{uu}(\omega_j)\delta_0(\omega_j+\omega_k)d\omega_1\ldots d\omega_{2n} \quad (9)$$

where $\prod_{j,k}^{2n}$ is a product over $2n!/n!2^n$ sets of n unordered pairs of the integers $1,\ldots,2n$. For example, for $n = 2$, there are 3 sets of 2 unordered pairs: $\{(1,2),(3,4)\}$; $\{(1,3),(2,4)\}$ and $\{(1,4),(2,3)\}$. The sum $\sum_p$ is performed over the products.[2] The operator $[.]_{sym}$ denotes the symmetrization of the kernel. More explicit expressions are possible but complex, since the product $H(\omega_1,\ldots,\omega_n)\cdot H(\omega_{n+1},\ldots,\omega_{2n})$ is not symmetric in general (Bedrosian and Rice, 1971).

E.g. the output power of a 3^{rd} order Volterra Series is:

$$\begin{aligned}
P(y(t)) &= E\left(\left(\sum_{k=0}^{3}\mathcal{H}^{(k)}[u(t)]\right)^2\right) = \left(H^{(0)}\right)^2 \\
&+ 2H^{(0)}\int_{-\infty}^{\infty} H^{(2)}(\omega,-\omega)S_{uu}(\omega)d\omega + \int_{-\infty}^{\infty}|H^{(1)}(\omega)|^2 S_{uu}(\omega)d\omega \\
&+ 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|H^{(2)}(\omega_1,\omega_2)|^2\cdot\prod_{i=1,2}S_{uu}(\omega_i)d\omega_i \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H^{(2)}(\omega_1,-\omega_1)H^{(2)}(\omega_2,-\omega_2)\cdot\prod_{i=1,2}S_{uu}(\omega_i)d\omega_i \\
&+ 6\iiint_{\infty}^{\infty}|H^{(3)}(\omega_1,\omega_2,\omega_3)|^2\cdot\prod_{i=1,2,3}S_{uu}(\omega_i)d\omega_i \\
&+ 9\iiint_{-\infty}^{\infty}H^{(3)}(\omega_1,-\omega_1,\omega_2)H^{(3)}(-\omega_2,\omega_3,-\omega_3)\cdot\prod_{i=1,2,3}S_{uu}(\omega_i)d\omega_i \\
&+ 6\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H^{(1)}(\omega_1)\cdot H^{(3)}(-\omega_1,\omega_2,-\omega_2)\cdot\prod_{i=1,2}S_{uu}(\omega_i)d\omega_i
\end{aligned} \quad (10)$$

For zero-mean Gaussian input, the expected value of the output signal is (Carassale and Kareem, 2010):

$$E(y(t)) = \sum_{\substack{k=0\\k\ even}}^{n}\frac{k!}{(k/2)!2^{k/2}}\int_{\substack{\boldsymbol{\Omega}_k\in\mathcal{R}^k\\ \boldsymbol{D}_k\boldsymbol{\Omega}_k=\boldsymbol{0}}} H^{(k)}(\boldsymbol{\Omega}_k)\prod_{r=1}^{k/2}S_{uu}(\omega_r)d\boldsymbol{\Omega}_k$$

where the summation is over even terms and $\boldsymbol{D}_k = [I_{k/2}, I_{k/2}]$, where $I_{k/2}$ is the identity matrix of size $k/2$.

The expanded version of $E(y(t))$ and $E(y(t))^2$ for a Volterra series of order 3 are

$$E\left(\sum_{k=0}^{3}\mathcal{H}^{(k)}[u(t)]\right) = H^{(0)} + \int_{-\infty}^{\infty}H^{(2)}(\omega,-\omega)S_{uu}(\omega)d\omega$$

$$\begin{aligned}
P_{DC}(y(t)) &= E\left(\sum_{k=0}^{3}\mathcal{H}^{(k)}[u(t)]\right)^2 = \left(H^{(0)}\right)^2 \\
&+ 2H^{(0)}\int_{-\infty}^{\infty}H^{(2)}(\omega,-\omega)S_{uu}(\omega)d\omega \\
&+ \iint_{-\infty}^{\infty}H^{(2)}(\omega_1,-\omega_1)H^{(2)}(\omega_2,-\omega_2)S_{uu}(\omega_1)S_{uu}(\omega_2)d\omega_1 d\omega_2
\end{aligned} \quad (11)$$

The calculation of the variance σ_y^2 using Eq. (8) is performed by subtracting Eq. (10) from Eq. (11). which leads to a cancellation of terms. As examples, Table 1 summarizes the calculation of σ_y^2 under zero mean Gaussian inputs with variance σ_u^2, for different kernel combinations.

[2] For more details see Eq.(92) by Rugh (1981).

Kernel Orders	**Volterra Series**	$\sigma_y^2 = \boldsymbol{E(y^2(t)) - [E(y(t))]^2}$
$\{0,1\}$	$h^{(0)} + \int_{-\infty}^{\infty}h^{(1)}(\tau)u(t-\tau)d\tau$	$\sigma_u^2\int_{-\infty}^{\infty}\lvert H^{(1)}(\omega)\rvert^2 d\omega$
$\{0,2\}$	$h^{(0)} + \int_{-\infty}^{\infty}h^{(2)}(\tau_1,\tau_2)u(t-\tau_1)u(t-\tau_2)d\tau_1 d\tau_2$	$2\sigma_u^4\iint_{-\infty}^{\infty}\lvert H^{(2)}(\omega_1,\omega_2)\rvert^2 d\omega_1 d\omega_2$
$\{0,3\}$	$h^{(0)} + \int_{-\infty}^{\infty}h^{(3)}(\tau_1,\tau_2,\tau_3)u(t-\tau_1)u(t-\tau_2)u(t-\tau_3)d\tau_1 d\tau_2 d\tau_3$	$6\sigma_u^6\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\lvert H^{(3)}(\omega_1,\omega_2,\omega_3)\rvert^2 d\omega_1 d\omega_2 d\omega_3$ $+9\sigma_u^6\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H^{(3)}(\omega_1,-\omega_1,\omega_2)\cdot H^{(3)}(-\omega_2,\omega_3,-\omega_3)d\omega_1 d\omega_2 d\omega_3$
$\{0,1,2\}$	$h^{(0)} + \int_{-\infty}^{\infty}h^{(1)}(\tau)u(t-\tau)d\tau$ $+\int_{-\infty}^{\infty}h^{(2)}(\tau_1,\tau_2)u(t-\tau_1)u(t-\tau_2)d\tau_1 d\tau_2$	$\sigma_u^2\int_{-\infty}^{\infty}\lvert H^{(1)}(\omega)\rvert^2 d\omega + 2\sigma_u^4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\lvert H^{(2)}(\omega_1,\omega_2)\rvert^2 d\omega_1 d\omega_2$
$\{0,1,3\}$	$h^{(0)} + \int_{-\infty}^{\infty}h^{(1)}(\tau)u(t-\tau)d\tau$ $+\int_{-\infty}^{\infty}h^{(3)}(\tau_1,\tau_2,\tau_3)u(t-\tau_1)u(t-\tau_2)u(t-\tau_3)d\tau_1 d\tau_2 d\tau_3$	$\sigma_u^2\int_{-\infty}^{\infty}\lvert H^{(1)}(\omega)\rvert^2 d\omega$ $+6\sigma_u^6\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\lvert H^{(3)}(\omega_1,\omega_2,\omega_3)\rvert^2 d\omega_1 d\omega_2 d\omega_3$ $+9\sigma_u^6\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H^{(3)}(\omega_1,-\omega_1,\omega_2)\cdot H^{(3)}(-\omega_2,\omega_3,-\omega_3)d\omega_1 d\omega_2 d\omega_3$ $+6\sigma_u^4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H^{(1)}(\omega_1)\cdot H^{(3)}(-\omega_1,\omega_2,-\omega_2)d\omega_1 d\omega_2$
$\{0,1,2,3\}$	$h^{(0)} + \int_{-\infty}^{\infty}h^{(1)}(\tau)u(t-\tau)d\tau$ $+\int_{-\infty}^{\infty}h^{(2)}(\tau_1,\tau_2)u(t-\tau_1)u(t-\tau_2)d\tau_1 d\tau_2$ $+\int_{-\infty}^{\infty}h^{(3)}(\tau_1,\tau_2,\tau_3)u(t-\tau_1)u(t-\tau_2)u(t-\tau_3)d\tau_1 d\tau_2 d\tau_3$	$\sigma_u^2\int_{-\infty}^{\infty}\lvert H^{(1)}(\omega)\rvert^2 d\omega + 2\sigma_u^4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\lvert H^{(2)}(\omega_1,\omega_2)\rvert^2 d\omega_1 d\omega_2$ $+6\sigma_u^6\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\lvert H^{(3)}(\omega_1,\omega_2,\omega_3)\rvert^2 d\omega_1 d\omega_2 d\omega_3$ $+9\sigma_u^6\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H^{(3)}(\omega_1,-\omega_1,\omega_2)\cdot H^{(3)}(-\omega_2,\omega_3,-\omega_3)d\omega_1 d\omega_2 d\omega_3$ $+6\sigma_u^4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H^{(1)}(\omega_1)\cdot H^{(3)}(-\omega_1,\omega_2,-\omega_2)d\omega_1 d\omega_2$

Table 1. Volterra series expansions and their corresponding output variance.

3.3 Definition of nonlinear Prediction Error Index Array.

Assume a multivariable system with m outputs and n inputs, where the i-th output y_i is represented by:

$$y_i(t) = \mathcal{H}_i^{(0)} + \sum_{j=1}^{n} \sum_{k=1}^{\infty} \mathcal{H}_{ij}^{(k)}[u_j(t)] \qquad (12)$$

where $\mathcal{H}_{ij}^{(k)}$ is the k-th order Volterra operator from the j-th input u_j to y_i. The term $\mathcal{H}^{(0)}$ is a constant output independent of the input, and the rest of the terms are:

$$\mathcal{H}_{ij}^{(k)}[u_j(t)] = \int_{\boldsymbol{\tau}_k \in \mathcal{R}^k} h_{ij}^{(k)}(\boldsymbol{\tau}_k) \prod_{r=1}^{k} u_j(t-\tau_r) d\boldsymbol{\tau}_k \ , \ (k = 1, 2, \dots)$$

The contribution of $u_j(t)$ to $y_i(t)$ is denoted as $y_{i,j}(t)$:

$$y_{i,j}(t) = \sum_{k=1}^{\infty} \mathcal{H}_{ij}^{(k)}[u_j(t)]$$

The PEIA is defined as:

$$[PEIA]_{ij} \triangleq \frac{\sigma^2(y_{i,j}(t))}{\sum_{i=1}^{m} \sigma^2(y_i(t))} = \frac{\sigma^2\left(\sum_{k=1}^{\infty} \mathcal{H}_{ij}^{(k)}[u(t)]\right)}{\sum_{i=1}^{m}\sum_{j=1}^{n} \sigma^2\left(\sum_{k=1}^{\infty} \mathcal{H}_{ij}^{(k)}[u_j(t)]\right)}$$

Example 2. A Wiener system is considered, with

$$y_i(t) = \sum_{j=1}^{3} \left(\int_{-\infty}^{\infty} g_{ij}(\tau) \cdot u_j(t-\tau) \cdot d\tau\right)^2 \qquad (13)$$

where $g_{ij}(\tau)$ is the impulse response of the single-input-single-output linear systems in Eq. (4).

The contribution $y_{i,j}(t)$ of input $u_j(t)$ to output $y_i(t)$ is:

$$y_{i,j}(t) = \left(\int_{-\infty}^{\infty} g_{ij}(\tau) u_j(t-\tau) d\tau\right)^2 = \iint_{-\infty}^{\infty} \underbrace{\prod_{k=\{1,2\}} g_{ij}(\tau_k)}_{h_{ij}^{(2)}(\tau_1,\tau_2)} u_j(t-\tau_k) d\tau_k$$

which shows that each output y_i is given by a Volterra series with second order kernels $h_{ij}^{(2)}(\tau_1, \tau_2)$ and the VFRF:

$$H_{ij}^{(2)}(\boldsymbol{j}\omega_1, \boldsymbol{j}\omega_2) = \iint_{-\infty}^{\infty} \prod_{k=\{1,2\}} e^{-\boldsymbol{j}\omega_k \tau_k} g_{ij}(\tau_k) d\tau_k = G_{ij}(\boldsymbol{j}\omega_1) \cdot G_{ij}(\boldsymbol{j}\omega_2)$$

To determine PEIA, we need to calculate the 2-dimensional integrals of the squared magnitude of the VFRF for each of the second order kernels as indicated by the second row in Table 1. As an example, we derive analytically the integral of the squared magnitude of the VFRF for a first order system with a quadratic linear output. That is, for the subsystems $(i, j) = (1, 3)$ and $(i, j) = (3, 1)$:

$$\sigma^2(y_{i,j}(t)) = \sigma^2\left(\left(\int_{-\infty}^{\infty} \frac{K_{ij}}{T_{ij}} e^{-\tau/T_{ij}} \cdot u(t-\tau) d\tau\right)^2\right)$$

$$= 2\sigma_{u_j}^4 \iint \left|\frac{K_{ij}}{(1+T_{ij}\omega_1)} \cdot \frac{K_{ij}}{(1+T_{ij}\omega_2)}\right|^2 d\omega_1 d\omega_2 = 2\sigma_{u_j}^4 K_{ij}^4 \pi^2 / T_{ij}^2$$

$$(i, j) = \{(1, 3), (3, 1)\}$$

$$K_{13} = -0.5/4,\ K_{31} = 0.5/2,\ T_{13} = 1/4,\ T_{31} = 1/2$$

A similar calculation for rest of the VFRFs leads too:

$$PEIA = \begin{pmatrix} \underline{0.3163} & 0.0065 & 0.0028 \\ 0.0032 & \underline{0.3690} & 0.0831 \\ 0.0111 & 0.0198 & \underline{0.1883} \end{pmatrix}$$

$$PEIA_{\ 11} + PEIA_{\ 22} + PEIA_{\ 33} = 0.8736$$

which means that the structurally reduced model composed the diagonal input-output channels has a variance of the prediction error of approximately 13% with zero-mean Gaussian excitation.

4. COMPARISON WITH LINEAR INTERACTION MEASURES

Consider a MIMO nonlinear system represented by Eq. (12). A small signal linearization would lead to:

$$y_i(t) = \mathcal{H}_i^{(0)} + \sum_{j=1}^{n} \mathcal{H}_{ij}^{(1)}[u_j(t)] = \mathcal{H}_i^{(0)} + \sum_{j=1}^{n} \int_{-\infty}^{\infty} h_{ij}^{(1)}(\tau) \cdot u_j(t-\tau) \cdot d\tau$$

Lemma 3. The value of the nonlinear PEIA when the variance of the excitation signal tends to 0 is equal to the value of PEIA for the small signal linearization.

Proof:

$$\lim_{\sigma_u \to 0} [PEIA]_{ij} = \lim_{\sigma_u \to 0} \frac{\sigma^2\left(\sum_{k=1}^{\infty} \mathcal{H}_{ij}^{(k)}[u_j(t)]\right)}{\sum_{i=1}^{m}\sum_{j=1}^{n} \sigma^2\left(\sum_{k=1}^{\infty} \mathcal{H}_{ij}^{(k)}[u_j(t)]\right)}$$

$$= \lim_{\sigma_u \to 0} \frac{\sigma_{u_j}^2 \int_{-\infty}^{\infty} |H_{ij}^{(1)}(\omega)|^2 d\omega \dots\dots}{\sum_{k,l=1}^{m,n} \sigma_u^2 \int_{-\infty}^{\infty} |H_{kl}^{(1)}(\omega)|^2 d\omega \dots\dots}$$

$$\frac{+2\sigma_{u_j}^4 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |H_{ij}^{(2)}(\omega_1, \omega_2)|^2 d\omega_1 d\omega_2 + \dots}{+2\sigma_u^4 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |H_{kl}^{(2)}(\omega_1, \omega_2)|^2 d\omega_1 d\omega_2 + \dots}$$

$$= \frac{\int_{-\infty}^{\infty} |H_{ij}^{(1)}(\omega)|^2 d\omega}{\sum_{k,l=1}^{m,n} \int_{-\infty}^{\infty} |H_{kl}^{(1)}(\omega)|^2 d\omega} = \frac{\|H_{ij}^{(1)}\|_2^2}{\sum_{k,l=1}^{m,n} \|H_{kl}^{(1)}\|_2^2}$$

■ **Example** 3.
Assume an output nonlinearity such that:

$$y_1(t) = \underbrace{x_{11}(t) + x_{11}(t)^2 + x_{11}(t)^3}_{y_{1,1}} + \underbrace{x_{12}(t)}_{y_{1,2}} + \underbrace{x_{13}(t) + x_{13}(t)^2 + x_{13}(t)^3}_{y_{1,3}}$$

$$y_2(t) = \underbrace{x_{21}(t) + x_{21}(t)^2}_{y_{2,1}} + \underbrace{x_{22}(t) + x_{22}(t)^3}_{y_{2,2}} + \underbrace{x_{23}(t) + x_{23}(t)^2}_{y_{2,3}}$$

$$y_3(t) = \underbrace{x_{31}(t) + x_{31}(t)^3}_{y_{3,1}} + \underbrace{x_{32}(t) + x_{32}(t)^2 + x_{32}(t)^3}_{y_{3,2}} + \underbrace{x_{33}(t) + x_{33}(t)^3}_{y_{3,3}}$$

where $x_{ij}(t)$ is the output of the subsystem G_{ij} in Eq. (4).

An approximation for small signals around $u_0 = [0, 0, 0]^T$ leads to the linear model in Eq. (4).

Applying other linear gramian-based IMs result in:

$$PM = \begin{pmatrix} \underline{0.4175} & 0.0124 & 0.0045 \\ 0.0157 & \underline{0.3122} & 0.0249 \\ 0.0181 & 0.0482 & \underline{0.1465} \end{pmatrix} ; \quad HIIA = \begin{pmatrix} \underline{0.2801} & 0.0405 & 0.0295 \\ 0.0489 & \underline{0.2291} & 0.0519 \\ 0.0590 & 0.0949 & \underline{0.1662} \end{pmatrix}$$

$$PM_{11} + PM_{22} + PM_{33} = 0.8762\ ;\ HIIA_{11} + HIIA_{22} + HIIA_{33} = 0.6754$$

The three gramian-based IMs indicate the diagonal configuration as the most adequate decentralized configuration.

The RGA for the linearized model is:

$$RGA = \begin{pmatrix} \underline{0.9680} & -0.0081 & 0.0401 \\ -0.0206 & \underline{1.0425} & -0.0220 \\ 0.0526 & -0.0344 & \underline{0.9819} \end{pmatrix}$$

The RGA is only applicable for the design of decentralized configurations, being the preferred input-output pairings those with values close to 1. The RGA indicates the same decentralized configuration as the gramian-based IMs.

For the calculation of the nonlinear PEIA, each of the outputs y_i can be represented by a Volterra series as

$$\mathcal{H}_{ij}^{(k)}[u_j(t)] = \int_{\boldsymbol{\tau}_k \in \mathcal{R}^k} \underbrace{\prod_{r=1}^{k} g_{ij}(\tau_r)}_{h_{ij}^{(k)}(\boldsymbol{\tau}_k)} \prod_{r=1}^{k} u_j(t-\tau_r) d\boldsymbol{\tau}_k$$

Using Table 1 derives in the following simplifications for the input-output channels with three Kernels:

$$\sigma_{y_{i,j}}^2 = \sigma_{u_j}^2 \int_{-\infty}^{\infty} |G_{ij}(\mathbf{j}\omega)|^2 d\omega + 8\sigma_{u_j}^4 \left(\int_{-\infty}^{\infty} |G_{ij}(\mathbf{j}\omega)|^2 d\omega \right)^2 + 15\sigma^6 \left(\int_{-\infty}^{\infty} |G_{ij}(\mathbf{j}\omega)|^2 d\omega \right)^3, for\ (i,j) \in \{(1,1),(1,3),(3,2)\}$$

Calculating the nonlinear PEIA for $\sigma_u^2 = 10^{-5}$, leads to the same result obtained for the linear case in Eq. (5), which validates that the calculation of the nonlinear PEIA for small signals converges to the value of PEIA for the linearization. Using a rigorous threshold of 0.7 for the contribution of the structurally reduced model leads to an sparse configuration including the following input-output channels $\{(1,1),(2,2),(3,3),(2,3)\}$.

An increase of the input variance to 0.005 results in:

$$PEIA_{\,0.005} = \begin{pmatrix} \underline{0.2497} & 0.0330 & 0.0217 \\ 0.0231 & \underline{0.2658} & 0.1179 \\ 0.0437 & 0.0587 & \underline{0.1863} \end{pmatrix}$$

$$[PEIA_{\,0.005}]_{11} + [PEIA_{\,0.005}]_{22} + [PEIA_{\,0.005}]_{33} = 0.7019$$

This increase in the output variance leads to operating conditions with a different control configuration. There is an increase in the diagonal dominance of the system and a decentralized diagonal controller is suggested.

Increasing the input variance to 0.36 leads to:

$$PEIA_{\,0.36} = \begin{pmatrix} \underline{0.3694} & 0.0034 & 0.0037 \\ 0.0027 & \underline{0.4059} & 0.0122 \\ 0.0096 & 0.0180 & \underline{0.1752} \end{pmatrix}$$

$$[PEIA_{\,0.36}]_{11} + [PEIA_{\,0.36}]_{22} + [PEIA_{\,0.36}]_{33} = 0.9504$$

which indicates that a diagonal controller is expected to behave almost as three independent SISO loops.

In this example, different excitation level results in different significance of the input-output channels, due to a different contribution of the nonlinearities. A higher level of excitation can be understood as a wider desired operating range.

5. CONCLUSIONS

A new Interaction Measure for Control Configuration Selection is introduced under the name Prediction Error Index Array (PEIA), which aids in the selection of a structurally reduced model for the design of a closed loop system. It has been shown that the sum of the values of the PEIA of the neglected input-output channels is equal to the variance (power) of the prediction error.

Using the Volterra series approach, the PEIA could be extended to nonlinear systems with the same properties. The most compelling property of the suggested method is that the indications for nonlinear systems converge to the ones for the linearized case, when the operating range becomes narrow and close to the operating point, used for the linearization. Moreover, it is also shown that wider operating ranges of a nonlinear system may render different control configurations than for narrow operating ranges. In that case it could also be observed that the recommeneded control configuration might have less complexity.

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